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Projections and Functions of Nash Equilibria

Yehuda John Levy^{*†}

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Abstract

We show that any non-empty compact semi-algebraic subset of mixed action profiles on a fixed player set can be represented as the projection of the set of equilibria of a game in which additional binary players have been added. Even stronger, we show that any semi-algebraic continuous function, or even any semi-algebraic upper-semicontinuous correspondence with non-empty convex values, from a bounded semi-algebraic set to the unit cube can be represented as the projection of an equilibrium correspondence of a game with binary players in which payoffs depend on parameters from the domain of the function or correspondence in a multi-affine way. Some extensions are also presented.

Keywords: Nash Equilibrium, Structure Theorem, Semialgebraic Geometry

JEL Classifications: C62, C65, C72

This work is dedicated to the memory of John Nash, who passed on during revision of this manuscript.

1 Introduction

As Nash equilibrium is the most fundamental solution concept in game theory, questions about the structure of Nash equilibria have received much attention. It is clear that, given a finite collection of players and action spaces, not every (non-empty, compact) set of mixed action profiles can arise as the set of Nash equilibria of a game on these players. Hence, it is natural to question which sets *can* arise.

^{*}Nuffield College, University of Oxford, UK, OX1 1NF.

[†]The author is very grateful to Guillaume Vigeral and Yannick Viossat for discussions on the topic - see Section 5.1 for a discussion on their work and how it compares to ours; to Xavier Venel and Ron Peretz for their remarks; to Yakov Babichenko for bringing to my attention the relation between my work and that on computational complexity; to Dominik Karos for proofreading some sections; and to Abraham Neyman for his excellent introductory paper on semi-algebraic sets with applications to game theory,[15], without which the author would undoubtedly have significantly less comprehension on the subject; and to two anonymous referees for their suggestions and corrections.

Nash equilibrium was defined by Nash (1950), [13], [14]. In the standard non-cooperative framework, Nash equilibria are those profiles of actions, in the mixed extension of the game (that is, in the extension in which players are allowed to use randomised strategies), against which no player has an incentive to unilaterally deviate. Nash equilibria are always guaranteed to exist (when the player and action spaces are finite), and it follows easily that the set of Nash equilibria of a game is always compact. In addition, since Nash equilibria are formally defined in terms of polynomial inequalities, the set of equilibria is *semi-algebraic*.

It is clear from observing the cases of only one or two players that, once the set of players is fixed, not every semi-algebraic, non-empty, compact subset of mixed action profiles can be the set of Nash equilibria of some game. Even if the player set is larger, once it has been fixed, algebraic techniques can give bounds on the number of components it may possess, or even on the ‘complexity’ of the individual components; the set of equilibria also must satisfy a multi-convexity property; see Section 2.4. Hence, a notable vein in the literature has been to study the *topological* and/or *algebraic* structures of the set of equilibria. In particular, it is known that every compact connected semi-algebraic set is *homeomorphic* to a connected component of the set of Nash equilibria of some game;¹ see [1] and the references within. Datta had already shown [7] that any algebraic variety (i.e., a set defined via polynomial equalities) is stably isomorphic to the set of completely mixed equilibria of a 3-player game, where this isomorphism notion allows for semi-algebraic homeomorphisms and equivalences of the form $V \times \mathbb{R}^K \sim V$. Bubelis [4] also gives a method of studying equilibria in general games by studying those of 3-player games; see Section 5.4. However, these results leave open questions on whether perhaps any compact semi-algebraic set can arise *precisely* - and not just up to topological or algebraic equivalence - in some way in the universe of Nash equilibria.

In this paper we present such a way. More specifically, given a non-empty compact semi-algebraic set X of mixed action profiles, we show that one can enlarge the player set by adding finitely many binary players, and define a game G on the larger player set, such that the projection of the set of equilibria of G to the actions of the original players is precisely X . As it turns out, at the same time of our research, an almost identical question was researched via different techniques by Guillaume Vigeral and Yannick Viossat, [19]. We contrast our work to theirs in Section 5.1; we mention at this point that their techniques do allow them to derive a bound on the number of additional binary players required as a function of the set.

For the purpose of our main result (and in contrast to [19]), we actually show a stronger result, which generalizes a strain of work from the literature on the computational complexity of computing Nash equilibria. A central step in

¹Semi-algebraic sets always possess finitely many connected components.

the seminal paper of [6] is "constructing games that perform simple arithmetical operations on mixed strategies", [6, p. 215]. In this paper, we show that any semi-algebraic continuous function from a bounded semi-algebraic set to the unit cube can be represented as the projection of an equilibrium correspondence of a game with binary players in which payoffs depend on parameters from the function's domain in a multi-affine way. We also generalize this result to upper semi-continuous semi-algebraic correspondences with convex non-empty values.

Section 2 presents the model of games and equilibria, the notion of semi-algebraic sets and some discussion on the restrictions on the set of Nash equilibria. The results are stated in Section 3, along with some discussion and examples, and the proofs are given in Section 4. Section 5 discusses some extensions and variations.

2 Games, Algebra, and Equilibria

2.1 Games

For a finite set of players I , with action spaces $(A^i)_{i \in I}$, a game is a mapping $G : \prod_{i \in I} A^i \rightarrow \mathbb{R}^I$ which assigns to each action profile a payoff for each player. G extends multi-affinely to action profiles $z \in \prod_{i \in I} \Delta(A^i)$, where $\Delta(A^i)$ denotes the simplex of probability distributions on A^i , by

$$G(z) = \sum_{a=(a^i)_{i \in I} \in \prod_{i \in I} A^i} \left(\prod_{i \in I} z^i[a^i] \right) G(a)$$

We introduce the following notion which will be very useful for us: For $N \in \mathbb{N}$, an \mathbb{R}^N -parametrized game $G\cdot$ on a set of players I with action spaces $(A^i)_{i \in I}$ is a game whose payoffs depend on a parameter $x = (x_1, \dots, x_N)$ in a multi-affine way: I.e., for each action profile $a \in \prod_{i \in I} A^i$, each $1 \leq k \leq N$, and each $(x_j)_{j \neq k} = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_N) \in \mathbb{R}^{N-1}$ the mapping $x \rightarrow G[x_1, \dots, x_{k-1}, x, x_{k+1}, \dots, x_N](a)$ is affine, of the form $a \cdot x + b$ for some $a, b \in \mathbb{R}^I$.

To understand a bit the meaning of an \mathbb{R}^N -parametrized game, by denoting $G_t = G[t]$ for each $t \in \{0, 1\}^N$, we see that we can express

$$G[x] = \sum_{t \in \{0, 1\}^N} \left(\prod_{k, t_k=1} x_k \prod_{k, t_k=0} (1 - x_k) \right) G_t \quad (2.1)$$

Hence, for $x \in [0, 1]^N$, one can view the game $G[x]$ as the expected game facing the players as a result of the following process: There are 2^N games, each for one sequence of bits in $\{0, 1\}^N$. Nature chooses the N bits independently, the i -th bit with probability $(x_i, 1 - x_i)$, and the players simultaneously have to choose their actions; their payoff is then assigned according to the game Nature chose

and the actions played. Alternatively but similarly, instead of Nature, we could have N binary players (in addition to the original I players) who are indifferent between their two actions; hence, any action profile for these N can result in equilibrium, and when they play the profile $\otimes_{j=1}^N (x_j, 1 - x_j)$, the expected game facing the players I is $G[x_1, \dots, x_N]$.

We adopt several conventions:

If $J \subseteq I$ is a subset of players, $G^J(z)$ denotes the payoffs to the players in J , and z^J denotes the mixed actions of the players in J ; formally, $G^J(z) = (G^i(z))_{i \in J}$, $z^J = (z^i)_{i \in J}$.

A *binary player* is a player with two actions, which we think of as '1' and '0', and instead of writing a mixed action as $(p, 1 - p)$, we denote the mixed action by the single number $p \in [0, 1]$, the probability of '1'. Similarly, if $x = (x_1, \dots, x_I) \in \mathbb{R}^I$, we view x as a mixed action profile of binary players I , and similarly sets $X \subseteq [0, 1]^I$ are viewed as sets of mixed action profiles.

2.2 Semi-Algebraic Sets and Functions

Let $\mathbb{R}[x_1, \dots, x_N]$ denote the ring² of polynomials in N variables, x_1, \dots, x_N . A semi-algebraic subset of \mathbb{R}^N is a set of the form

$$\cup_{j=1}^m \cap_{i=1}^{m_j} \{(x_1, \dots, x_N) \in \mathbb{R}^n \mid P_{i,j}(x) *_{i,j} 0\} \quad (2.2)$$

for some finite collection $(P_{i,j})_{i,j} \subseteq \mathbb{R}[x_1, \dots, x_N]$, where for each i, j , $*_{i,j}$ is one of the relations $>, <, \geq, \leq, =, \neq$. The semi-algebraic sets form an *algebra*: I.e., they are closed under finite unions, finite intersections, and complements.

Equivalently (e.g., [3, Ch. 2]), semi-algebraic sets are those that can be expressed as a formula in first-order logic whose atoms are of the form $P(x) > 0$ or of the form $P(x) = 0$ for some $P \in \mathbb{R}[x_1, \dots, x_N]$. In particular, we mention the Tarski-Seidenberg theorem:

Theorem 2.1. *Let $A \subseteq \mathbb{R}^N$ be semi-algebraic, let $\pi_K : \mathbb{R}^N \rightarrow \mathbb{R}^K$ denote the projection to a subset $K \subseteq \{1, \dots, N\}$ of coordinates. Then $\pi_K(A)$ is semi-algebraic.*

A *semi-algebraic function* $f : A \rightarrow \mathbb{R}^K$, where $A \subseteq \mathbb{R}^N$, is one whose graph $Gr(f) := \{(x, y) \in A \times \mathbb{R}^K \mid y = f(x)\}$ is semi-algebraic: It follows from Theorem 2.1 that the domain A is semi-algebraic, and that the image / inverse image of a semi-algebraic set under a semi-algebraic function is also semi-algebraic; it also follows that the composition of semi-algebraic functions is semi-algebraic. A *correspondence*, denoted $F : A \rightrightarrows \mathbb{R}^K$, assigns to each

²A ring is an algebraic structure with operations of addition and multiplication satisfying certain axioms; we will not need to make use of the specific axioms, which can be found in any introductory text on abstract algebra.

$x \in A$ a subset $F(x) \subseteq \mathbb{R}^K$; a correspondence is semi-algebraic if its graph $Gr(F) := \{(x, y) \in A \times \mathbb{R}^K \mid y \in F(x)\}$ is semi-algebraic. Recall also that $F : A \Rightarrow \mathbb{R}^K$ is called *upper semi-continuous* if $Gr(F)$ is closed in $A \times \mathbb{R}^K$.

2.3 Nash Equilibria

The Nash equilibria of a game G are those $z \in \prod_{i \in I} \Delta(A^i)$ satisfying

$$G^j(z) \geq G^j(b, z^{-j}), \quad \forall j \in I, b \in A^j$$

where $z^{-j} = (z_i)_{i \neq j}$. It is easy to see that the set of Nash equilibria of a game G with action sets $(A^i)_{i \in I}$ is a compact semi-algebraic set; indeed, it is the collection of $z \in \prod_{i \in I} \mathbb{R}^{A^i}$, such that:

$$z^j[b] \geq 0, \quad \forall j \in I, b \in A^j$$

$$\sum_{b \in A^j} z^j[b] = 1, \quad \forall j \in I$$

$$\sum_{a \in \prod_{i \in I} (A^i)_{i \in I}} \left(\prod_{i \in I} z^i[a^i] \right) G^j(a) \geq \sum_{a \in \prod_{i \in I} (A^i)_{i \in I}, a^j = b} \left(\prod_{i \in I, i \neq j} z^i[a^i] \right) G^j(a), \quad \forall j \in I, b \in A^j$$

It's easy to see similarly, using (2.1), that if $G[\cdot]$ is an \mathbb{R}^N -parametrized game, then the correspondence $E_G : \mathbb{R}^N \Rightarrow \prod_{i \in I} \Delta(A^i)$, where $E_G(x)$ is the Nash equilibria of $G[x]$, is semi-algebraic and upper semi-continuous.

2.4 Sets Which Are Not Sets of Equilibria

To motivate our results, we first discuss limitations to the 'complexity' the set of Nash equilibria can have (given a collection of players).

Given a finite set of players I with finite action spaces $(A^i)_{i \in I}$, not every compact semi-algebraic subset X of $\prod_{i \in I} \Delta(A^i)$ can be the set of equilibria E of some game. For example, if I consists of a single player, then E must be the convex hull of pure strategies. If I consists of two players, it can be shown that E must be the finite union of products of the form $S_1 \times S_2$ with S_j being a convex polytype in $\Delta(A_j)$, [8]; one can also observe that the set of Nash equilibria in two-player games must be bi-convex.³

Even for more players, even though the set of Nash equilibria can be somewhat richer (e.g., [4], [7]) not every compact semi-algebraic subset of the space of mixed actions need be the set of equilibria of some game. Like in the two-player case, the set of equilibria satisfies a multi-convexity property: If x, y are

³That is, if $(x, y), (x, z)$ are equilibria and $\alpha \in [0, 1]$, then $(x, \alpha y + (1 - \alpha)z)$ is also an equilibrium, and similarly w.r.t. the first coordinate. Hence, for example, in a 2×2 game, the set of profiles in which at least one player plays pure cannot be the set of equilibria of any game. I am grateful to an anonymous referee for pointing this out.

two equilibria profiles that differ only in the action of a single player, then any convex combination is an equilibrium as well. To place further restrictions on the set of equilibria, we recall:

Proposition 2.1. *There is a function $\phi : \mathbb{N}^3 \rightarrow \mathbb{N}$ such that the set of solutions of r polynomial equalities and inequalities in N variables of degrees⁴ at most $d \geq 2$ has at most $\phi(r, N, d)$ connected components.*

Although we will not need it, we remark that a crude bound is $\phi(r, N, d) = d(2d - 1)^{N+r-1}$, [5]; a better bound of $r^N \cdot O(d)^N$ is given in [18].

In particular, we deduce from Section 2.3, since the set of Nash equilibria are defined via $|I| + 2 \sum_{j \in I} |A^j|$ polynomial inequalities and equalities, of degrees at most I , and is a subset of $\mathbb{R}^{\sum_{j \in I} |A^j|}$, that:

Corollary 2.2. *There is a function $\psi : \mathbb{N}^2 \rightarrow \mathbb{N}$ such that the number of connected components of the set of Nash equilibria of any I -player game in which Player $k \in I$ has m_k pure strategies is at most $\psi(|I|, \sum_{k \in I} m_k)$.*

We can also always find *connected* sets which *cannot* be the set (or even a component) of the set of Nash equilibria for the players I with action spaces $(A^i)_{i \in I}$. Again applying Proposition 2.1, we can deduce:

Corollary 2.3. *There is a function $\psi' : \mathbb{N}^2 \rightarrow \mathbb{N}$ such that if E is the set of Nash equilibria of any I -player game in which Player $k \in I$ has m_k pure strategies, and P is an affine space of co-dimension 1,⁵ then $E \cap P$ has at most $\psi'(|I|, \sum_{k \in I} m_k)$.*

Indeed, the restriction to P requires adding a single additional equality.

Hence, for example, for $k \in \mathbb{N}$, define the function $f_k : [0, 1] \rightarrow [0, 1]$ by

$$f_k(x) = 2k \cdot \min\{|x - \frac{1}{k} \cdot n| \mid n \in \mathbb{Z}\}$$

and let $L = \{(x, y) \in \mathbb{R}^2 \mid y = \frac{1}{2}\}$. (See Figure 1.) Let $V : \mathbb{R}^2 \rightarrow \prod_{i \in I} \mathbb{R}^{A^i}$ be an injective affine map such that $V([0, 1]^2) \subseteq \prod_{i \in I} \Delta(A^i)$. Then for $k > \frac{1}{2}\psi'(|I|, (|A^i|)_{i \in I})$, $V(Gr(f_k))$ cannot be the set of equilibria of any game, since $V(Gr(f_k)) \cap P$ has $2k$ components (all singletons) for any affine space P such that $V(\mathbb{R}^2) \cap P = V(L)$.

3 Results

The main result of this paper is:

⁴The degree of a monomial is the sum of the degrees from all variables; e.g., the degree of x^3y^2z is 6.

⁵That is, $P \subseteq \prod_{i \in I} \mathbb{R}^{A^i}$ is of the form $u + V$, where V is a linear space of dimension one less than the space $\prod_{i \in I} \mathbb{R}^{A^i}$.

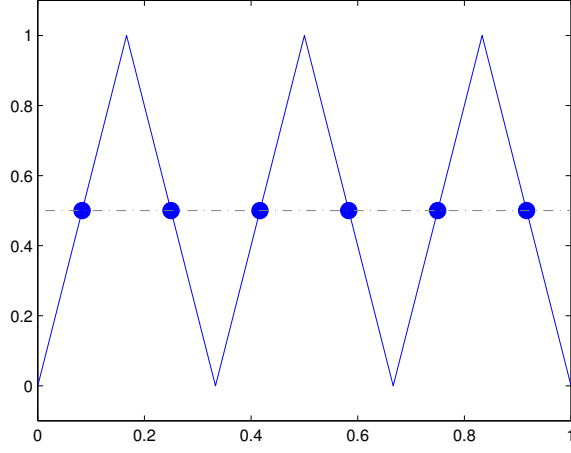


Figure 1: The Function f_3 with the line $L = \{(x, y) \mid y = \frac{1}{2}\}$.

Theorem 3.1. *Let I be a finite set of players with finite sets $(A^i)_{i \in I}$ of actions, and let $\emptyset \neq X \subseteq \prod_i \Delta(A^i)$ be compact and semi-algebraic. Then there exists a set of binary players J , and a game G on the player set $I \cup J$ such the projection of the set of equilibria of G to $\prod_i A^i$ is X ; i.e.,*

$$X = \{(z^i)_{i \in I} \mid z \in \prod_{i \in I} \Delta(A^i) \times \prod_{j \in J} \Delta(\{1, 0\}) \text{ is an equilibrium of } G\}$$

From the examples and arguments in Section 2.4, we deduce that the size of the set of additional binary players J in Theorem 3.1 cannot be bounded as a function only of I and the $(A^i)_{i \in I}$, but may be arbitrary large as a function of the given compact semi-algebraic set X (even when X is connected). Our proofs, in attempt to keep things simple and with the focus on using Theorem 3.2 below as our main tool, have not made an attempts to derive a bound on the size of J as a function of X (e.g., on the number of polynomials needed to define X). The parallel work of [19], which employs different techniques, gives an alternative proof of Theorem 3.1 for the case that $(A^i)_{i \in I}$ are binary,⁶ and does give a bound as a function of the set X , which we discuss in Section 5.1.

Note also that the projection of the set of equilibria must be semi-algebraic by Theorem 2.1, as well clearly as compact. Hence, clearly the conclusion of Theorem 3.1 can not be strengthened to give more general sets.

⁶This is not a serious restriction, as one can then deduce the general case using Lemma 4.11 below.

Although there are many questions for further research that could be posed, we raise one at this point: Can the construction in Theorem 3.1 be done such that the projection from the equilibria of G to X is injective? Even for relatively simple sets (e.g., a line segment with a non-differentiable 'kink' in it), we do not know.

As mentioned above, the main step in proving Theorem 3.1 is the following theorem, which is of interest in itself.

Theorem 3.2. *Let $A \subseteq \mathbb{R}^N$ be bounded and semi-algebraic; and let $f : A \rightarrow [0, 1]^K$ be a continuous semi-algebraic function. Then there exists an \mathbb{R}^N -parametrized game $G\cdot$ on a set of binary players $\{\alpha_1, \dots, \alpha_K\} \cup J$ such that for each $x \in A$, in any equilibrium z of $G[x]$, we have $(z^{\alpha_1}, \dots, z^{\alpha_K}) = f(x)$.*

In other words, any semi-algebraic function on a bounded semi-algebraic set can be realised as the projection of the equilibrium correspondence of a game with binary players in which payoffs depend multi-affinely on coordinates from the function's domain. As mentioned in the introduction, [6, Sec. 4] already discusses such techniques when considering functions which represent arithmetic operations.⁷ We remark that the theorem does not imply that the equilibrium of $G[x](\cdot)$ is unique; only that the projection of the equilibria to the players $\alpha_1, \dots, \alpha_K$ is uniquely determined by x .

Theorem 3.2 was proven by the author in [12] for the case $A = [0, 1]$, $K = 1$, and f which is piece-wise linear. We remark that the semi-algebraicity and the continuity is necessary by Theorem 2.1 and since the equilibrium correspondence is upper-semicontinuous.

We will in fact strengthen Theorem 3.2 (although this is not needed for the proof of Theorem 3.1):

Theorem 3.3. *Let $A \subseteq \mathbb{R}^N$ be bounded and semi-algebraic, and let $F : A \Rightarrow [0, 1]^K$ be an upper semi-continuous semi-algebraic correspondence with non-empty convex values (i.e., $\forall x \in A$, $F(x) \neq \emptyset$ and is convex). Then there exists a \mathbb{R}^N -parametrized game $G[\cdot]$ on a set of binary players $\{\alpha_1, \dots, \alpha_K\} \cup J$ such that for each $x \in A$,*

$$F(x) = \{(z^{\alpha_1}, \dots, z^{\alpha_K}) \mid z \text{ is an equilibrium of } G[x]\} \quad (3.1)$$

In other words, any semi-algebraic u.s.c. correspondence with non-empty convex values on a bounded semi-algebraic set can be realised as the projection

⁷To quote, for example, from p. 214 there, "...if a player v has two pure strategies, say 0 and 1, then every mixed strategy of that player corresponds to a real number $p[v] \in [0, 1]$ which is precisely the probability that the player plays strategy 1. Identifying players with these numbers, we are interested in constructing games that perform simple arithmetical operations on mixed strategies; for example, we are interested in constructing a game with two input players $v1$ and $v2$ and another output player $v3$ so that in any Nash equilibrium the latter plays the sum of the former, i.e., $p[v3] = \min(p[v1] + p[v2], 1)$.

of the equilibrium correspondence of a game with binary players in which pay-offs depend multi-affinely on coordinates from the function's domain. Clearly, by the same reasoning above, the upper-semicontinuity and semi-algebraicity are necessary.

We remark that this theorem is not true if the convexity assumption is dropped. Let $F : [0, 1] \Rightarrow [0, 1]$ be defined by

$$F(x) = \begin{cases} \{0\} & \text{if } x < \frac{1}{2} \\ \{0, 1\} & \text{if } x = \frac{1}{2} \\ \{1\} & \text{if } x > \frac{1}{2} \end{cases}$$

Suppose the conclusion of the theorem held for this F with an \mathbb{R} -parametrized game $G[\cdot]$ on a set of players $\{\alpha\} \cup J$. Then there is a connected component C of the set of equilibria of $G[\frac{1}{2}]$ which is *essential*, i.e., for any neighbourhood V of C , there is a neighbourhood U of $\frac{1}{2}$ such that for $y \in U$, $G[y]$ contains equilibria in V ; see [9], or [10] for the related notion of *stability*. The projection of C is connected. Hence, for any $\varepsilon > 0$ there is $\delta > 0$ such that for any x_1, x_2 , with $\frac{1}{2} - \delta < x_1 < \frac{1}{2} < x_2 < \frac{1}{2} + \delta$, $G[x_1], G[x_2]$ either must both contain equilibria y with $y^\alpha < \varepsilon$, or must both contain equilibria y with $y^\alpha > 1 - \varepsilon$.

It appears that the convexity assumption may be weakened somewhat, but it is left for future research to determine the full class of correspondences for which the result holds. (E.g., what if F is contractible-valued?)

3.1 Examples of Theorem 3.1

In this section we present three simple examples of Theorem 3.1. The first introduces a useful component which will be used in the proofs. The third demonstrates the technique used in the proof of Proposition 4.3.

Example #1: The Identity Function (Extended) Observe the following useful \mathbb{R} -parametrized two-player game (for $q = \frac{1}{2}$, this is the game *matching pennies*):

$$H[q] = \begin{array}{|c|c|} \hline -1, 1 & 4q - 1, 4q - 3 \\ \hline 3 - 4q, 1 - 4q & -1, 1 \\ \hline \end{array} \quad (3.2)$$

For $0 < q < 1$, the unique equilibrium is $(q, 1 - q) \otimes (q, 1 - q)$; for $q \in \{0, 1\}$, the set of equilibria is $\{(q, 1 - q) \otimes (w, 1 - w) \mid w \in [0, 1]\}$; for $q > 1$ (resp. $q < 0$) the unique equilibrium is $(1, 0) \times (0, 1)$ (resp. $(0, 1) \times (1, 0)$). Hence, denoting

$$u(x) = \min[\max[x, 0], 1] = \begin{cases} 1 & \text{if } x \geq 1 \\ x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } x \leq 0 \end{cases} \quad (3.3)$$

then for any $q \in \mathbb{R}$ and equilibrium (z^α, z^β) of $H[q]$, $z^\alpha = u(q)$. A game that gives a similar representation of the identity function in $[0, 1]$ follows from

Lemma 4.11.

Example #2: Addition. Let $+$: $[0, \frac{1}{2}] \times [0, \frac{1}{2}] \rightarrow [0, 1]$ be the addition function. Define the \mathbb{R}^2 -parametrized two-player game by $G_+[x, y] := H[x + y]$, where H is defined by (3.2). Again, for any $x, y \in [0, \frac{1}{2}]$, and equilibrium (z^α, z^β) of $G[x, y]$, $z^\alpha = x + y$.

Example #3: Define $f : [0, \frac{\sqrt{2}}{2}] \times [0, \frac{\sqrt{2}}{2}] \rightarrow [0, 1]$ by $f(x, y) = x^2 + y^2$. Define a \mathbb{R}^2 -parametrized 6-player game $G\cdot$ with binary players $(\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha, \beta)$ in the following way:

$$\begin{aligned} G^{\alpha_1, \beta_1}[x, y](u) &= H[x](u^{\alpha_1}, u^{\beta_1}) \\ G^{\alpha_2, \beta_2}[x, y](u) &= H[y](u^{\alpha_2}, u^{\beta_2}) \\ G^{\alpha, \beta}[x, y](u) &= H[u^{\alpha_1} \cdot u^{\beta_1} + u^{\alpha_2} \cdot u^{\beta_2}](u^\alpha, u^\beta) \end{aligned}$$

where recall that u^p denotes the mixed action - represented as $u^p \in [0, 1]$ - of Player p . Since $G^{\alpha, \beta}$ depends affinity on each other player's action, $G^{\alpha, \beta}$ is well-defined. If z is an equilibrium of $G[x, y](\cdot)$, then

$$z^{\alpha_1} = z^{\beta_1} = x, \quad z^{\alpha_2} = z^{\beta_2} = y$$

Therefore, for mixed action profile v of α, β , we have

$$G^{\alpha, \beta}[x, y](v, z^{-\{\alpha, \beta\}}) = H[x^2 + y^2](v)$$

and therefore $z^\alpha = x^2 + y^2$.

4 Proofs

To simplify notation, if $\alpha_1, \dots, \alpha_n$ are players and J_1, \dots, J_k are sets of players, we write $(\alpha_1, \dots, \alpha_n, J_1, \dots, J_k)$ instead of $\{\alpha_1, \dots, \alpha_n\} \cup J_1 \cup \dots \cup J_k$.

4.1 Proof of Theorem 3.2

To facilitate the proof, we introduce the following terminology:

Definition 4.1. A function $f : A \rightarrow [0, 1]$ will be called *exactly representable* if it satisfies the conclusion of Theorem 3.2; i.e., there is an \mathbb{R}^N -parametrized game $G_f[\cdot]$ on a set of binary players $\{\alpha_f\} \cup J_f$ such that for any $x \in A$ and any equilibrium z of $G_f[x]$, $z^{\alpha_f} = f(x)$. It will be called *representable* if for some $a, b \in \mathbb{R}$ with $a \neq 0$, $a \cdot f + b$ is exactly representable.

Theorem 3.2 states in particular (the case $K = 1$) that every continuous semi-algebraic function from a bounded semi-algebraic set to $[0, 1]$ is exactly representable.

4.1.1 Representable Functions

This section is dedicated to showing that the family of representable functions is an algebra lattice (i.e., a vector space closed under multiplication and taking of point-wise maxima/minima) which is closed under composition and contains all polynomials.

Corollary 4.2. *If $D \subseteq \mathbb{R}^N$ and $\Lambda : \mathbb{R}^N \rightarrow \mathbb{R}$ is affine in each coordinate,⁸ and $\Lambda(D)$ is bounded, then Λ is representable. If $\Lambda(D) \subseteq [0, 1]$, then Λ is exactly representable.*

Proof. Let $a \neq 0, b \in \mathbb{R}$ such that $a \cdot \Lambda(D) + b \subseteq [0, 1]$. (If $\Lambda(D) \subseteq [0, 1]$, take $a = 1, b = 0$.) Letting H be the \mathbb{R} -parametrized game on two binary players which represents the identity in $[0, 1]$ in Example #1 of Section 3.1, we define

$$G_\Lambda[x_1, \dots, x_N](\cdot) = H[\Lambda(x_1, \dots, x_N)](\cdot)$$

which clearly obeys the required multi-affine conditions, and gives the required game as $0 \leq a\Lambda(D) + b \leq 1$; if $x \in D$ and (z^α, z^β) is an equilibrium of $G_\Lambda[x](\cdot)$, then $z^\alpha = a\Lambda(x) + b$. \square

Proposition 4.3. *The composition of (exactly) representable functions is (exactly) representable.*

Proof. Let $a_1 \cdot f_1 + b_1, \dots, a_K \cdot f_K + b_K, a_g \cdot g + b_g$ ($a_1, \dots, a_K, a_g \neq 0$) be exactly represented by \mathbb{R}^N -parametrized games $G_1[\cdot], \dots, G_K[\cdot]$ and an \mathbb{R}^K -parametrized game $G_g[\cdot]$, on player sets (α_j, J_j) , $j = 1, \dots, K$, and (α_g, J_g) , respectively. (If f_1, \dots, f_K, g are exactly representable, take $a_1, \dots, a_K, a_g \equiv 1, b_1, \dots, b_K, b_g \equiv 0$.) Define G on the set of players $(\alpha_g, (\alpha_j)_j, J_g, (J_j)_j)$ by

$$G^{\alpha_j, J_j}[x](z) = G_j[x](z^{\alpha_j, J_j}), \quad j = 1, \dots, K$$

i.e., (α_j, J_j) play as in G_j , while

$$G^{\alpha_g, J_g}[x](z) = G_g\left[\frac{1}{a_1}(z^{\alpha_1} - b_1), \dots, \frac{1}{a_K}(z^{\alpha_K} - b_K)\right](z^{\alpha_g, J_g}) \quad (4.1)$$

Clearly $G[\cdot]$ represents $a_g \cdot g \circ (f_1, \dots, f_K) + b_g$ (via the player α_g), since in any equilibrium z of $G[x]$, $\forall j, z^{\alpha_j} = a_j f_j(x) + b_j$, and

$$z^{\alpha_g} = a_g \cdot g\left(\frac{1}{a_1}(z^{\alpha_1} - b_1), \dots, \frac{1}{a_K}(z^{\alpha_K} - b_K)\right) + b_g$$

\square

The following corollaries follow easily:

Corollary 4.4. *A representable function f in A is exactly representable iff $0 \leq f \leq 1$ in A .*

⁸I.e., if $1 \leq k \leq N$, and $(x_j)_{j \neq k} \in \mathbb{R}^{N-1}$, then the map $\mathbb{R} \rightarrow \mathbb{R}$ given by $x \rightarrow \Lambda(x_1, \dots, x_{j-1}, x, x_{j+1}, \dots, x_N)$ is affine.

Corollary 4.5. *The sum, difference, and product of finitely many representable functions in a set A is representable in A .*

Note that A need not be bounded in these last two corollaries.

Corollary 4.6. *Every polynomial is representable on any bounded set.*

Proposition 4.7. *The functions $\mathbb{R}^N \rightarrow \mathbb{R}$ given by $(x_1, \dots, x_N) \rightarrow \min[x_1, \dots, x_N]$ and $\rightarrow \max[x_1, \dots, x_N]$ are representable on any bounded subset of \mathbb{R}^N .*

Proof. By Corollary 4.5 and Proposition 4.3, it suffices to show that the function $(x, y) \rightarrow \max[x, y]$ is representable in any bounded set $D \times D$. Letting $M = \sup_{x \in D} |x|$, this follows from the above results and Example #1 of Section 3.1, which establishes the representability of the function u given by (3.3), and the observation

$$\max[x, y] = M \cdot \max\left[\frac{1}{M}(x - y), 0\right] + y = M \cdot \left(u\left(\frac{1}{M}(x - y)\right)\right) + y$$

□

4.1.2 Proof of Theorem 3.2 (For $K = 1$)

Proposition 4.8. *Let $\emptyset \neq B \subseteq \mathbb{R}^N$ be an open semi-algebraic set. Then there is a continuous function $\psi : \mathbb{R}^N \rightarrow \mathbb{R}$ which is representable on any bounded subset of \mathbb{R}^N , and such that $\psi(x) > 0$ if $x \in B$ and $\psi(x) = 0$ if $x \notin B$.*

Proof. By the finiteness theorem (e.g., [3, 2.7.2]), since B is open, B can be written as⁹

$$B = \cup_{i=1}^n \cap_{j=1}^{m_i} \{x \in \mathbb{R}^N \mid P_{i,j}(x) > 0\}$$

for some polynomials $(P_{i,j})_{i=1, \dots, n; j=1, \dots, m_i}$. Define $\phi : \mathbb{R}^N \rightarrow \mathbb{R}$ by

$$\phi(x) = \max \left\{ \min\{P_{i,j}(x) \mid j = 1, \dots, m_i\} \mid i = 1, \dots, n \right\}$$

and then $\psi : \mathbb{R}^N \rightarrow \mathbb{R}$ by $\psi(x) = \max[\phi(x), 0]$. ψ has the desired properties, and is representable on any bounded set by Corollaries 4.6 and 4.7 and Proposition 4.3. □

Lemma 4.9. *Let $U \subseteq A \subseteq \mathbb{R}^N$ be semi-algebraic, with U relatively open in A . Then there is a semi-algebraic open set $V \subseteq \mathbb{R}^N$ with $V \cap A = U$.*

Proof. Denote $B = A \setminus U$, which is also semi-algebraic; hence, so is its closure (in \mathbb{R}^N), \overline{B} (e.g., [3, Prop. 2.2.2]). Since U is relatively open in A , $U \cap \overline{B} = \emptyset$; hence, $V := \mathbb{R}^N \setminus \overline{B}$ is open, semi-algebraic, and satisfies $V \cap A = U$. □

⁹A little bit of thought shows that the fact that every open semi-algebraic set can be written in this way - using only the strong inequalities - is not an obvious conclusion, although it may appear so at first glance.

Now we prove Theorem 3.2 for the case $K = 1$, i.e., $f : A \rightarrow [0, 1]$ with $A \subseteq \mathbb{R}^N$ being bounded and semi-algebraic. Define

$$B_+ = \{(x, y) \in A \times [0, 1] \mid y > f(x)\} \quad (4.2)$$

$$B_- = \{(x, y) \in A \times [0, 1] \mid y < f(x)\} \quad (4.3)$$

Note that B_\pm are semi-algebraic and, due to the continuity of f , are relatively open in $A \times [0, 1]$. Let $\psi_+, \psi_- : \mathbb{R}^N \times [0, 1] \rightarrow \mathbb{R}$ be representable in $A \times [0, 1]$ such that for each $x \in A$ and $y \in [0, 1]$, $\psi_+(x, y) > 0$ (resp. $\psi_-(x, y) > 0$) if $(x, y) \in B_+$ (resp. $\in B_-$), and $= 0$ otherwise. To obtain such functions, one applies Proposition 4.8 to open semi-algebraic sets whose intersection with $A \times [0, 1]$ are precisely B_+, B_- , respectively, which exist by Lemma 4.9. Since A is bounded, we may assume w.l.o.g. that $\psi_+, \psi_- < 1$ on $A \times [0, 1]$.

Let ψ_+ (resp. ψ_-) be exactly represented in $A \times [0, 1]$ by the \mathbb{R}^{N+1} -parameterized games G_+, G_- with players (α_+, J_+) (resp. (α_-, J_-)). Then define on $(\alpha_f, \alpha_+, J_+, \alpha_-, J_-)$ the \mathbb{R}^N -parametrized game $G\cdot$ given by:

$$G^{\alpha_+, J_+}[x_1, \dots, x_n](z) = G_+[x_1, \dots, x_n, z^{\alpha_f}](z^{\alpha_+, J_+})$$

$$G^{\alpha_-, J_-}[x_1, \dots, x_n](z) = G_-[x_1, \dots, x_n, z^{\alpha_f}](z^{\alpha_-, J_-})$$

$$G^{\alpha_f}[x_1, \dots, x_n](z) = -z^{\alpha_+} \cdot z^{\alpha_f} - z^{\alpha_-} \cdot (1 - z^{\alpha_f})$$

G then clearly represents f . (See Figure 2.) Indeed, let z be an equilibrium of $G[x](\cdot)$ for some $x \in A$. If $z^{\alpha_f} < f(x)$, then $(x, z^{\alpha_f}) \in B_-$ and $(x, z^{\alpha_f}) \notin B_+$, so (since z is an equilibrium) $z^{\alpha_-} = \psi_-(x, z^{\alpha_f}) > 0$ while $z^{\alpha_+} = \psi_+(x, z^{\alpha_f}) = 0$. Therefore, $-z^{\alpha_+} > -z^{\alpha_-}$, so (again, since z is an equilibrium) $z^{\alpha_f} = 1$, a contradiction. A similar contradiction is reached if $z^{\alpha_f} > f(x)$.

4.1.3 Proof of Theorem 3.2 (General Case)

Now suppose $f = (f_1, \dots, f_K)$. Informally, we represent each of the f_j on their own and then play them 'independently'. Formally, for each $1 \leq j \leq K$, there is an \mathbb{R}^N -parametrized game $G_j[\cdot]$ on the set of binary players (α_j, J_j) which exactly represents f_j . Then define the \mathbb{R}^N -parametrized game $G[\cdot]$ on the set of players $(\alpha_1, \dots, \alpha_K, J_1, \dots, J_K)$ by

$$G^{\alpha_j, J_j}[x](z) = G_j[x](z^{\alpha_j, J_j})$$

G is clearly the desired game.

4.2 Proof of Theorem 3.1

4.2.1 Proof of Theorem 3.1 (Binary Players)

First we assume the players in I are binary: $A^i = \{1, 0\}$ for all $i \in I$. Recall in such a case a mixed action profile of the players I is an element of $[0, 1]^I$. Let's

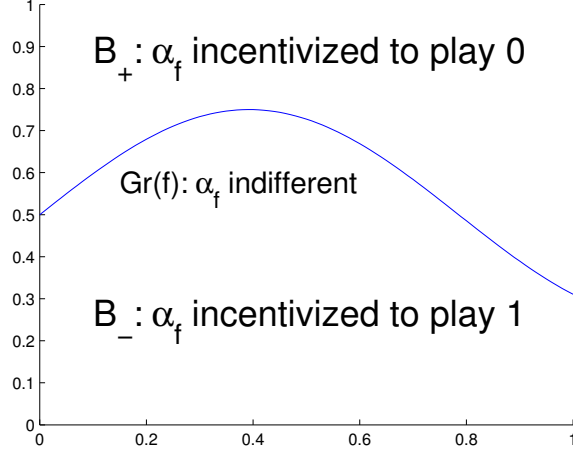


Figure 2: The sets B_+, B_- .

say a subset $X \subseteq [0, 1]^I$ is *representable* if it satisfies the conclusion of Theorem 3.1; i.e., there is a set of binary players J and a game G on the set of players $I \cup J$ such that the projection of the equilibria of G to \mathbb{R}^I is X . For brevity, for a game G , let $NE(G)$ be the set of equilibria of G .

- Proposition 4.10.** 1. *Finite products of representable sets are representable; i.e., if for $j = 1, \dots, M$, $X_j \subseteq [0, 1]^{m_j}$ is representable, then $\times_j X_j \subseteq [0, 1]^{\sum_j m_j}$ is representable.*
2. *If $X \subseteq [0, 1]^I$ is representable and $\psi : X \rightarrow [0, 1]^K$ is continuous and semi-algebraic, then $\text{image}(\psi) = \psi(X)$ is representable.*
3. *The sets $[0, 1]$ and $F := \{0, \frac{1}{2}, 1\}$ are representable.*

Proof. 1. Intuitively, we play the games which represent each of the sets independently. Formally: Let G_j be a game on a set of players $I_j \cup J_j$ with $X_j = \{z^{I_j} \mid z^{I_j, J_j} \in NE(G_j)\}$. Define G on a set of players $\cup_j (I_j \cup J_j)$ by $G^{I_j, J_j}(u) = G_j(u^{I_j, J_j})$. Then $\times_j X_j = \{z^{\cup_i I_i} \mid z^{\cup_j I_j \cup J_j} \in NE(G)\}$.

2. By Theorem 3.2, there is an \mathbb{R}^I -parametrized game $G_\psi[\cdot]$ on a set of players $K \cup J_\psi$ such that for each $x \in [0, 1]^I$ and any equilibrium z of $G_\psi[x]$, $z^K = \psi(x)$. There is also a game G_X on a set of players $I \cup J_X$ such that $X = \{z^I \mid z^{I \cup J_X} \in NE(G_X)\}$.

Define then the desired game G on a set of players $I \cup K \cup J_\psi \cup J_X$ by

$$G^{I \cup J_X}(u) = G_X(u^{I \cup J_X})$$

$$G^{K \cup J_\psi}(u) = G_\psi[u^I](u^{K \cup J_\psi})$$

3. The representability of $[0, 1]$ is obvious. For F , take a 2×2 symmetric coordination game. □

Using the well-known triangulation of semi-algebraic sets, e.g., [3, Ch. 9], it follows that for $N = \dim(X)$,¹⁰ X is the union of finitely many semi-algebraic continuous images of $[0, 1]^N$, and hence, for some $M \in \mathbb{N}$, there is a continuous semi-algebraic mapping $\psi : [0, 1]^N \times F^M \rightarrow [0, 1]^I$ such that $\text{image}(\psi) = X$.¹¹
¹² By parts (1) and (3) of Proposition 4.10, $[0, 1]^N \times F^M$ is representable; hence, by part (2) of Proposition 4.10, $X = \text{image}(\psi)$ is representable.

Hence, we have proved Theorem 3.1 when I consists of binary players.

4.2.2 Proof of Theorem 3.1 (General Players)

To move to the general case, we use:

Lemma 4.11. *Given a finite set B of actions, there is an \mathbb{R}^B -parametrized $(|B| + 1)$ -player game, $G_{\mathbb{N}}[x]$, where Player α has action set B and the players $(\beta^j)_{j \in B}$ are binary, such that for $x \in \Delta(B)$, any equilibrium¹³ z of $G_{\mathbb{N}}[x]$ satisfies $z^\alpha = x$.*

Proof. For each $b \in B$, $x = (x_b)_{b \in B} \in \mathbb{R}^B$ and action profile y , define

$$G_{\mathbb{N}}^{\beta_b}[x](1, y^{-\beta_b}) = y^\alpha[b]$$

$$G_{\mathbb{N}}^{\beta_b}[x](0, y^{-\beta_b}) = x_b$$

and

$$G_{\mathbb{N}}^\alpha[x](b, y^{-\alpha}) = \frac{1}{2} - y^{\beta_b}$$

Since $z^\alpha, x \in \Delta(B)$, if $z^\alpha \neq x$, then we must have some $b^* \in B$ with $z^\alpha[b^*] > x[b^*]$ and some $b^o \in B$ with $z^\alpha[b^o] < x[b^o]$; but then we would have

$$G_{\mathbb{N}}^{\beta_{b^*}}[x](1, z^{-\beta_{b^*}}) > G_{\mathbb{N}}^{\beta_{b^*}}[x](0, z^{-\beta_{b^*}})$$

$$G_{\mathbb{N}}^{\beta_{b^o}}[x](1, z^{-\beta_{b^o}}) < G_{\mathbb{N}}^{\beta_{b^o}}[x](0, z^{-\beta_{b^o}})$$

and therefore, since z is an equilibrium,

$$z^{\beta_{b^*}}[1] = z^{\beta_{b^o}}[0] = 1$$

¹⁰The notion of the dimension of a semi-algebraic set is well-defined; for our purposes, it is enough to know that there exists an N for which this decomposition is possible.

¹¹This is where we use the non-emptiness of X .

¹²We could have used a set F of two elements, which is also representable - but the advantage of three elements is that the equilibria of game representing can be taken to be stable; two equilibria is a degenerate situation.

¹³If x is completely mixed, the equilibrium is also unique: each β^j mixes equally.

so

$$G_{\mathbb{N}}^{\alpha}[x](b^*, z^{-\alpha}) = -\frac{1}{2} < \frac{1}{2} = G_{\mathbb{N}}^{\alpha}[x](b^o, z^{-\alpha})$$

Since z is an equilibrium, $z^{\alpha}[b^*] = 0 \leq x[b^*]$, a contradiction to $z^{\alpha}[b^*] > x[b^*]$. \square

Now, let I be a finite set of players with finite sets $(A^i)_{i \in I}$ of actions, and let $\emptyset \neq X \subseteq \prod_{i \in I} \Delta(A^i) \subseteq [0, 1]^{\cup_i A^i}$ be compact semi-algebraic. By the already established Theorem 3.1 for binary players, there is a game G on a set of binary players $\{\gamma_{i,b} \mid i \in I, b \in A^i\} \cup J$ such that

$$X = \{(z^p)_{p \in \{\gamma_{i,b} \mid i \in I, b \in A^i\}} \in \mathbb{R}^{\cup_i A^i} \mid z \text{ is an equilibrium of } G\} \quad (4.4)$$

Extend the game G by adding players $(\alpha_i, J_i)_{i \in I}$, where $J_i = \{\beta_{i,b} \mid b \in A^i\}$, the J_i being binary and α_i having action set I , and payoffs defined by

$$G^{\alpha_i, J_i}(u) = G_{\mathbb{N}}[(u^{\gamma_{i,b}})_{b \in A^i}](u^{\alpha_i, J_i})$$

with $G_{\mathbb{N}}$ defined in Lemma 4.11. G is then the desired game, as in any equilibrium z of G , $z^i = (z^{\gamma_{i,b}})_{b \in A^i}$, i.e., $z^i[b] = z^{\gamma_{i,b}}$ for $b \in B$; then apply (4.4).

4.3 Proof of Theorem 3.3

4.3.1 Proof of Theorem 3.3 (The Case $K = 1$)

This proof follows precisely as the proof of Theorem 3.2 (the case $K = 1$) in Section 4.1.2, except that the sets B_+, B_- are now defined by:

$$B_+ = \{(x, y) \in A \times [0, 1] \mid \forall t \in F(x), y > t\}$$

$$B_- = \{(x, y) \in A \times [0, 1] \mid \forall t \in F(x), y < t\}$$

Indeed, by the upper-semicontinuity of F on A , the sets B_+, B_- are relatively open in $A \times [0, 1]$; they are also semi-algebraic by Theorem 2.1, as B_+ is the complement in $A \times [0, 1]$ of the projection of the set

$$\{(x, y, t) \in A \times [0, 1] \times [0, 1] \mid (x, t) \in Gr(F), y \leq t\}$$

and similarly for B_- .

4.3.2 Proof of Theorem 3.3 (General Case)

Now we prove the general version of Theorem 3.3 - i.e., for range of arbitrary dimension K . We do so inductively; suppose $K \geq 2$ and we have proven for correspondences with range in $K - 1$ dimensions. Let $F : A \Rightarrow \mathbb{R}^K$ be as in Theorem 3.3, with $A \subseteq \mathbb{R}^N$ semi-algebraic and bounded.

Define the projection of $Gr(F)$ to the first $N + K - 1$ coordinates,

$$\Pi = \{(x, y_1, \dots, y_{K-1}) \in A \times [0, 1]^{K-1} \mid \exists y_K \in [0, 1] \text{ s.t. } (x, y_1, \dots, y_{K-1}, y_K) \in Gr(F)\}$$

By Theorem 2.1, Π is semi-algebraic; it is also easily seen (since F takes values in the compact set $[0, 1]^K$) to be bounded. Let $H : \Pi \Rightarrow [0, 1]$ be such that $Gr(H) = Gr(F)$, and by slightly abusive notation, let Π also denote the correspondence $\Pi : A \Rightarrow [0, 1]^{K-1}$ with graph $Gr(\Pi)$. H, Π are then semi-algebraic, and are seen to have non-empty convex values contained in $[0, 1], [0, 1]^{K-1}$ respectively.

By the induction hypothesis, there is an \mathbb{R}^{N+K-1} -parametrized game G_H on a set of binary players (α_K, J_H) such that for each $(x, y_1, \dots, y_{K-1}) \in \Pi$,

$$H(x, y_1, \dots, y_{K-1}) = \{z^{\alpha_K} \mid z \text{ is an equilibrium of } G_H[x, y_1, \dots, y_{K-1}]\}$$

and there is an \mathbb{R}^N -parametrized game G_Π on a set of players $(\alpha_1, \dots, \alpha_{K-1}, J_\Pi)$ such that for each $x \in A$,

$$\Pi(x, y_1, \dots, y_{K-1}) = \{(z^{\alpha_1}, \dots, z^{\alpha_{K-1}}) \mid z \text{ is an equilibrium of } G_\Pi[x]\}$$

Now, define the \mathbb{R}^N -parametrized game $G[\cdot]$ on the set of players $(\alpha_1, \dots, \alpha_{K-1}, \alpha_K, J_\Pi, J_H)$ by

$$G^{\alpha_1, \dots, \alpha_{K-1}, J_\Pi}[x](z) = G_\Pi[x](z^{\alpha_1}, \dots, z^{\alpha_{K-1}}, z^{J_\Pi})$$

$$G^{\alpha_K, J_H}[x](z) = G_H[x, z^{\alpha_1}, \dots, z^{\alpha_{K-1}}](z^{\alpha_K, J_H})$$

Clearly, if $x \in A$, z is an equilibrium of $G[x]$ iff z^{α_K, J_H} is an equilibrium of $G_H[x, z^{\alpha_1}, \dots, z^{\alpha_{K-1}}]$ and $(z^{\alpha_1}, \dots, z^{\alpha_{K-1}}, z^{J_\Pi})$ is an equilibrium of $G_\Pi[x]$, which can be iff $z^{\alpha_K} \in H(x, z^{\alpha_1}, \dots, z^{\alpha_{K-1}})$ and $(z^{\alpha_1}, \dots, z^{\alpha_{K-1}}) \in H(x)$, which is iff $(z^{\alpha_1}, \dots, z^{\alpha_{K-1}}, z^{\alpha_K}) \in H(x)$.

5 Extensions & Discussion

5.1 Bounds on Additional Players in Theorem 3.1 & Comparison to [19]

Our techniques do not give a bound on the size of the set of additional binary players J needed in Theorem 3.1 in order to define a game whose projection of equilibria onto the first I players coordinates is precisely X . One does see that given a function represented explicitly using only polynomials, composition of functions, and the maximum / minimum functions, it would not be difficult to follow the construction of Section 4.1.1 to give a bound on the number of players needed in the parametrized game used to represent the function in the sense of Theorem 3.2. However, there is a difficulty using this bound to deduce a bound for the number of additional players needed in Theorem 3.2 for more general functions, even given representation of the graph using polynomials equalities/inequalities, and there is a further difficulty to make the leap from a bound on additional players for functions in Theorem 3.2 to a bound on additional players for sets in Theorem 3.1:

In the first deduction, the problem arises that even given an explicit description of the graph of a function f as a semi-algebraic set, what we actually need in Theorem 3.2 is *not* this description but descriptions of the strict epigraph and strict hypograph of f (the sets B_+, B_- defined by (4.2) and (4.3) in Section 4.1.2) using *only strict* polynomial inequalities via the finiteness theorem; it is unclear to what extent this complexifies the representation. In the second deduction, a problem arises since the proof of Theorem 3.1 from Theorem 3.2 uses a triangulation and does so in an implicit, rather than explicitly constructive, way; it is also not clear to what extent this complicates matters.

It is possible that both difficulties could be overcome using more carefully techniques of computational geometry, but we have not attempted to do so.

The related work of [19] shows that in Theorem 3.1, if $(A^i)_{i \in I}$ are binary, and X can be written¹⁴

$$X = \cup_{n=1}^N \cap_{m=1}^M \{x \in \mathbb{R}^I \mid P_{n,m}(x) \geq 0\}$$

and d_{max} is a uniform bound on the maximal degree of any variable¹⁵ in the polynomials $(P_{n,m})$, then we can bound the number of additional players J by

$$|J| \leq 1 + M \cdot N + 2|I| \cdot (1 + \ln_2(d_{max}))$$

If one deduces our Theorem 3.1 from the version of Theorem 3.1 in which all action spaces are binary, via the argument used in Section 4.2.2, one can show that in the general case, if

$$X = \cup_{n=1}^N \cap_{m=1}^M \left\{ \prod_{i \in I} \mathbb{R}^{A^i} \mid P_{n,m}(x) \geq 0 \right\}$$

then

$$|J| \leq 1 + M \cdot N + \sum_{i \in I} |A^i| \cdot (3 + 2 \ln_2(d_{max})) + |I|$$

We remark that the work in [19] is different than ours in that the construction of the games there are very explicit, instead of going through results on functions à la Theorem 3.2. This allows them to tweak the results to derive other interesting conclusions, for example that if the set X of Theorem 3.1 contains only rational points, then the payoffs in the game can be chosen to be all integers (although it may require adding more players than otherwise needed).

5.2 Payoffs

It follows easily from Theorem 3.1 that:¹⁶

¹⁴ As in Section 4.1.2, such a representation exists due to the finiteness theorem.

¹⁵ For example, the maximal degree of $x^4 \cdot y^2$ is 4.

¹⁶ Our deduction of Theorem 5.1 from Theorem 3.1 is different from a parallel deduction done in [19].

Theorem 5.1. *Let $\emptyset \neq P \subseteq \mathbb{R}^N$ be a compact semi-algebraic set. Then there is $K \geq N$ and a K -player game G with binary players $\{\alpha_1, \dots, \alpha_K\}$ such that the projection of the set of equilibrium payoffs to G to the players $\{\alpha_1, \dots, \alpha_N\}$ is P ; precisely,*

$$P = \{(p^{\alpha_1}, \dots, p^{\alpha_N}) \mid (p^{\alpha_1}, \dots, p^{\alpha_K}) \text{ is an equilibrium payoff of } G\}$$

We remark that if $N \geq 3$ and one would allow the players in the game G to have arbitrarily large (but finite) action spaces, it is not clear that one cannot suffice with $K = N$. (If $N = 2$, [11] shows that the set of Nash equilibrium payoffs must be a finite union of rectangles.) This question is left for future research.

Proof. Let $a_1, \dots, a_N, b_1, \dots, b_N \in \mathbb{R}$, with $a_1, \dots, a_N \neq 0$, be such that for all $p = (p^1, \dots, p^N) \in P$ and each $1 \leq j \leq N$, $0 \leq a_j \cdot p^j + b_j \leq 1$. Define

$$Q = \{(a_j p^j + b_j)_{1 \leq j \leq N} \mid (p^1, \dots, p^N) \in P\}$$

By Theorem 3.1, there are sets of binary players I, J , where $I = \{\beta_1, \dots, \beta_N\}$, and a game G on the set of players $I \cup J$ such that

$$Q = \{(z^{\beta_1}, \dots, z^{\beta_N}) \mid z \in (\Delta(\{0, 1\}))^{I \cup J} \text{ is an equilibrium of } G\}$$

Now extend G to the set of players $\{\alpha_1, \dots, \alpha_N\} \cup I \cup J$ by

$$G^{\alpha_j}(y) = \frac{y^{\beta_j} - b_j}{a_j}$$

It's easy to see that this extension of G is the required game. \square

It follows in a similarly fashion from Theorem 3.3:

Theorem 5.2. *Let $A \subseteq \mathbb{R}^N$ be bounded and semi-algebraic, and let $F : A \Rightarrow \mathbb{R}^K$ be an upper semi-continuous semi-algebraic bounded¹⁷ correspondence with convex non-empty values.¹⁸ Then there exists an \mathbb{R}^N -parametrized game $G\cdot$ on a set of binary players $\{\alpha_1, \dots, \alpha_M\}$, for some $M \geq K$, such that for all $x \in A$,*

$$F(x) = \{(p^{\alpha_1}, \dots, p^{\alpha_K}) \mid (p^{\alpha_1}, \dots, p^{\alpha_M}) \text{ is an equilibrium payoff of } G[x]\}$$

5.3 Countably Many Players

If I is an infinite set of players¹⁹ with finite action spaces $(A^i)_{i \in I}$, we say that G is a game on the set of players I if $G : \prod_{i \in I} A^i \rightarrow \mathbb{R}^I$ is continuous in the

¹⁷I.e., $\cup_{x \in A} F(x)$ is bounded.

¹⁸In particular, F may be a bounded continuous semi-algebraic function.

¹⁹I am grateful to Xavier Venel for posing to me the question of what one could derive if one allowed for countable many players.

Tychonoff topology.²⁰ The payoffs extend naturally to mixed strategies.²¹ The continuity requirement is equivalent to that for every $i \in I$ and every $\varepsilon > 0$, there is a finite set $I_0 \subseteq I$ such that if x, y are two action profiles (mixed or pure) with $x^j = y^j$ for each $j \in I_0$, then $|G^i(x) - G^i(y)| < \varepsilon$.

Under this condition, [16] has shown that equilibrium (in mixed strategies) exists; it is easy to show that the set of equilibria is compact.

We can use countably many players to generalize Theorem 3.2 to *any* continuous function. The notion of \mathbb{R}^N -parametrized games extends to \mathbb{R}^N -parametrized games with countably many players and countably many parameters when one adds the requirement of continuity w.r.t. the Tychonoff topology *jointly* on action profiles *and* on parameters.

Theorem 5.3. *Let $K, M \in \mathbb{N} \cup \{\infty\}$, $A \subseteq \mathbb{R}^K$ be compact and let $f : A \rightarrow [0, 1]^M$ be continuous. Then there exists an \mathbb{R}^K -parametrized game G on a countable set of binary players $\{\alpha_j\}_{j=1}^M \cup J$ such that for each $x \in A$, in any equilibrium z of $G[x]$, we have $(z^{\alpha_j})_{j=1}^M = f(x)$.*

Proof. As in Section 4.1.3, it suffices to treat the case $M = 1$; call such a function *exactly countably representable* on A , or ECR.

Lemma 5.1. *The family of ECR functions on A is closed under uniform limits.*

Proof. We sketch a proof:

- Using techniques as in Section 4.1.1, one shows that the linear space spanned by the ECR functions on A is a vector lattice closed under compositions, and that Corollary 4.2 extends to exact countable representability for $\Lambda : \mathbb{R}^N \rightarrow [0, 1]$ when one requires Λ to be continuous.
- A standard argument, e.g. [17, Thm. 3.11],²² shows that if ψ is a uniform limit of ECR functions, it can be written as $\psi = \sum_{k=1}^{\infty} M_k \psi_k$, where $(M_k)_{k=1}^{\infty}$ is a sequence in \mathbb{R} satisfying $\sum_{k=1}^{\infty} |M_k| < \infty$, and each $0 \leq \psi_k \leq 1$ is ECR.
- Combining these, with $\Lambda(x_1, x_2, \dots) = \sum_{k=1}^{\infty} M_k x_k$, shows that if $(\psi_k)_{k=1}^{\infty}$ are ECR on A , $\psi = \sum_{k=1}^{\infty} M_k \psi_k$ is ECR on A .

□

Theorem 5.3 now follows: By the Stone-Weierstrauss approximation theorem,²³ for each $k \in \mathbb{N}$, we can find a polynomial²⁴ p_k satisfying $0 \leq p_k \leq 1$ in A

²⁰This is equivalent to requiring that each player's payoff is continuous in the Tychonoff topology.

²¹The extension is well-defined because the assumed continuity w.r.t. the Tychonoff topology implies Borel measurability and boundedness.

²²In particular, the decomposition $f = \max[f, 0] - \max[-f, 0]$ is useful to guarantee that we can have $\psi_k \geq 0$ for all k .

²³It is here that the compactness of the domain is used.

²⁴When $K = \infty$, we emphasise this is a polynomial in finitely many coordinates.

(which is clearly ECR by Theorem 3.2) and such that $\sup_{x \in A} |p_k(x) - f(x)| \leq \frac{1}{k}$. \square

Theorem 3.1 also generalizes:

Theorem 5.4. *Let I be a finite or infinitely countable set of players with finite action spaces $(A^i)_{i \in I}$. Let $\emptyset \neq X \subseteq \prod_{i \in I} \Delta(A^i)$ be compact. Then there exists a countable set of binary players J and a game G on the set of players $I \cup J$ such that*

$$X = \{(z^i)_{i \in I} \mid z \in \prod_{i \in I} \Delta(A^i) \times \prod_{j \in J} \Delta(\{1, 0\}) \text{ is an equilibrium of } G\}$$

It suffices to prove the case in which the players I are binary; a profile of mixed actions is then an element of $[0, 1]^I$. The general case follows by the same argument used in Section 4.2.2.

Like in Section 4.2.1, call a set which satisfies the conclusion of Theorem 5.4 *countably representable*. The parallel of Proposition 4.10, with almost identical proof (except using Theorem 5.3 instead of Theorem 3.2), shows:

Proposition 5.2. *1. Countable products of countably representable sets are countably representable.*

2. If $X \subseteq [0, 1]^{\mathbb{N}}$ is countably representable and $\psi : X \rightarrow [0, 1]^I$ is continuous, then $\text{image}(\psi) = \psi(X)$ is countably representable.

3. $F := \{0, \frac{1}{2}, 1\}$ is countably representable.

It is well-known, since X is compact and metrizable, that there is a continuous surjective function $\psi : F^{\mathbb{N}} \rightarrow X$, e.g. [2]. Hence it follows by Proposition 5.2 that any compact $\emptyset \neq X \subseteq \mathbb{R}^I$ is countably representable.

5.3.1 An Example

Take the set

$$A = \{x, y \in [0, 1]^2 \mid x = 0 \text{ or } \exists n \in \mathbb{N}, y = n \cdot x\}$$

(See Figure 3.) A is clearly compact but not semi-algebraic; if it were, then $A \setminus \{0\}$ would be semi-algebraic but with infinitely many connected components, an impossibility, e.g., [3, Thm. 2.4.4]. Now, define the following game G :

- The players are α, β, ζ and countably many pairs $(\gamma_j, \delta_j)_{j \in \mathbb{N}}$. The players are binary; actions are $\{0, 1\}$.
- Each pair (γ_j, δ_j) plays a game which has two pure equilibria $(1, 1)$ and $(0, 0)$, e.g.,

1, 1	0, 0
0, 0	0, 0

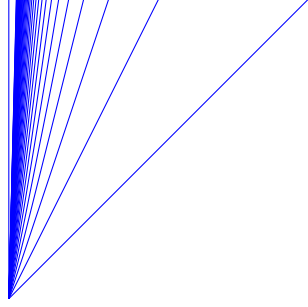


Figure 3: The Set A of the Example

- ζ is indifferent; $G^\zeta \equiv 0$.
- For a pure action profile u^γ of the players $(\gamma^j)_{j \in \mathbb{N}}$, denote

$$n(u^\gamma) = \min\{j \in \mathbb{N} \mid u^{\gamma_j} = 1\}$$

where $\min \emptyset = \infty$.

- For pure profile u ,

$$G^{\alpha, \beta}(u) = H[u^\zeta / n(u^\gamma)](u^\alpha, u^\beta)$$

where H was defined in (3.2) and $\frac{*}{\infty} = 0$.

These payoffs are easily checked to be continuous. Let $(x, y) \in A$. Then there is an equilibrium z of the game with $(x, y) = (z^\alpha, z^\zeta)$: Indeed, write $x = \frac{y}{n}$ (we may have $n = \infty$). Define $z^\zeta = y$, $z^\alpha = z^\beta = y/n$. Let $z^{\gamma_j} = z^{\delta_j} = 1$ if $j = n$ and $= 0$ otherwise. z is easily seen to be an equilibrium. Conversely, if z is an equilibrium, $(z^{\gamma_j}, z^{\delta_j})_{j \in \mathbb{N}}$ are pure, and $z^\alpha = z^\zeta / n(z)$, so $(z^\alpha, z^\zeta) \in A$.

5.4 Theorem 3.1 With Non-Binary Additional Players

In [4], a scheme for understanding equilibria of general games through equilibria of 3-player games is presented. We will use this method to prove the following variation of Theorem 3.1:

Theorem 5.5. *Let I be a finite set of players with finite sets $(A^i)_{i \in I}$ of actions, and let $\emptyset \neq X \subseteq \prod_i \Delta(A^i)$ be compact and semi-algebraic. Then there exists a game G on a player set $I \cup \{\mu, \nu, \eta\}$, where μ, ν, η have some finite action spaces A^μ, A^ν, A^η , such that the projection of the equilibria of G to $\prod_i A^i$ is X ; more precisely,*

$$X = \{(z^i)_{i \in I} \mid z \in \prod_{i \in I \cup \{\mu, \nu, \eta\}} \Delta(A^i) \text{ is an equilibrium of } G\}$$

Hence, it is possible to add only 3 additional players; however, their action spaces can be arbitrarily large.²⁵

It is not clear if this result can be improved to two additional players; however, it follows from the following proposition (and the discussion in Section 2.4) that this result can *not* be improved to a single additional player:

Proposition 5.3. *Let $N \in \mathbb{N}$, and let A^1, \dots, A^N, A^{N+1} be finite action spaces. Denote $K = 1 + (N+1) \times \prod_{j=1}^N |A^j|$. Then for any game G with $N+1$ players with action spaces A^1, \dots, A^N, A^{N+1} , there is a subset $B^{N+1} \subseteq A^{N+1}$ of size at most K such that*

$$\begin{aligned} & \{(z^j)_{j \leq N} \mid (z^1, \dots, z^N, z^{N+1}) \text{ is an equilibrium of } G\} \\ &= \{(z^j)_{j \leq N} \mid (z^1, \dots, z^N, z^{N+1}) \text{ is an equilibrium of } G'\} \end{aligned}$$

where G' denotes the restriction of G to the action spaces A^1, \dots, A^N, B^{N+1} .

Proof. (Sketch) Each action of Player $N+1$ can be viewed as a function of profiles of the other players $\prod_{j=1}^N A^j$ to payoff profiles in \mathbb{R}^{N+1} , i.e., an element of $(\mathbb{R}^{N+1})^{\prod_{j=1}^N A^j}$. One then applies Caratheodory's theorem to represent each action as a convex combination of at most K actions. \square

In order to prove Theorem 5.5, we recall the terminology and a result from [4]. Let $(C^k)_{k \in K}, (D^m)_{m \in M}$ be finite action sets. Denote $\overline{C} = \prod_{k \in K} C^k$, $\overline{D} = \prod_{m \in M} D^m$. A *reduction scheme* consists of partial mappings²⁶ $\rho : \cup_{k \in K} C^k \rightarrow \cup_{m \in M} D^m$, $\phi : M \times \overline{D} \rightarrow \mathbb{R}$, $\psi : M \times \overline{D} \rightarrow K \times \overline{C}$, such that $\text{dom}(\phi), \text{dom}(\psi)$ - the domains of ϕ, ψ - form a partition of $M \times \overline{D}$. Denote $\hat{C}^k = \text{dom}(\rho) \cap C^k$.

Let x (resp. y) be a mixed action profile in $\prod_{k \in K} \Delta(C^k)$ (resp. $\prod_{m \in M} \Delta(D^m)$). We say that x is generated by y if²⁷

$$\sum_{\xi^k \in \hat{C}^k} y[\rho(\xi^k)] \neq 0 \text{ and } x^k[c^k] = \frac{y[\rho(c^k)]}{\sum_{\xi^k \in \hat{C}^k} y[\rho(\xi^k)]}, \forall k \in K, c^k \in \hat{C}^k \quad (5.1)$$

and

$$x^k[c_k] = 0 \quad \forall k \in K, c^k \in C^k \setminus \hat{C}^k$$

Clearly each mixed profile in $\prod_{m \in M} \Delta(D^m)$ generates at most one profile $\prod_{k \in K} \Delta(C^k)$. We say that a game G_C on the action sets $(C^k)_{k \in K}$ *reduces* to the game G_D on the action sets $(D^m)_{m \in M}$ via the reduction scheme (ρ, ϕ, ψ) if:

- If $(m, d) \in \text{dom}(\phi)$, then $G_D^m(d) = \phi(m, d) \in \mathbb{R}$.

²⁵Indeed, the discussion in Section 2.4 shows that we cannot hope to bound the sizes of their action spaces without considering the structure of X .

²⁶A partial mapping from U to V is a mapping from a subset of U to V .

²⁷If $d^m \in D^m \subseteq \cup_{m'} D^{m'}$, $y[d^m] = y^m[d^m]$; this is convenient as $\rho(c^k), \rho(c'^k)$ may belong to different action spaces even if c^k, c'^k are both actions of Player k .

- If $(m, d) \in \text{dom}(\psi)$, then $G_D^m(d) = G_C^k(c)$, where $(k, c) = \psi(m, d) \in K \times \bar{C}$.
- Any equilibrium of G_D generates an equilibrium of G_C .
- Any equilibrium of G_C is generated by an equilibrium of G_D .

Theorem 3 of [4] states:

Theorem 5.6. *Let $(C^k)_{k \in K}$ be fixed finite action spaces. Then there exists a reduction scheme by which any game on these action spaces reduces to a 3-player game.*

Remark 5.4. Note that in such a scheme, we must have $\hat{C}^k = C^k$ for each $k \in K$ - i.e., $\text{dom}(\rho) = \cup_{k \in K} C^k$ - since no strategy not in \hat{C}^k can be used in any equilibrium; and we must have ρ being injective, since if $i, j \in I$, $a^i \in A^i$, $b^j \in A^j$, $\rho(a^i) = \rho(b^j)$, then for any equilibrium x of any game on $\prod_k C^k$, we would have $x^i[a^i] > 0$ iff $x^j[b^j] > 0$.

We now construct a general technique, and deduce Theorem 5.5 easily from it. Let I, K, M be sets of players with action spaces $(A^i)_{i \in I}, (C^k)_{k \in K}, (D^m)_{m \in M}$. Let (ρ, ϕ, ψ) be a reduction scheme from $(C^k)_{k \in K}$ to $(D^m)_{m \in M}$ as above, with ρ being injective with domain $\cup_k C^k$, like the scheme guaranteed by Theorem 5.6 and Remark 5.4.

Let G_0 be a game on the players $I \cup K$. Define the $\mathbb{R}^{\cup_i A^i}$ -parametrized $G_0\cdot$ on the players K defined, for $x \in \prod_{i \in I} \Delta(A^i)$ by $G_0[x](\cdot) = G_0^K(x, \cdot)$ (i.e., the game induced for the players in K when the players I are restricted to playing x), and then extended multi-affinely to $\mathbb{R}^{\cup_i A^i}$.

Let \mathcal{S} be the map from games on $(C^k)_{k \in K}$ to games on $(D^m)_{m \in M}$ induced the reduction scheme (ρ, ϕ, ψ) . Define the game G on the players $I \cup M$ by:

$$G^M((y^i)_{i \in I}, (y^m)_{m \in M}) = (\mathcal{S} \circ (G_0[(y^i)_{i \in I}]))(y^m)_{m \in M} \quad (5.2)$$

which is well-defined as \mathcal{S} is affine, and

$$G^I[y^I, y^M] = G_0^I(y^I, x^K) \times \prod_{k \in C^k} \left(\sum_{\xi^k \in C^k} y[\rho(\xi^k)] \right) \quad (5.3)$$

where $x^K \in \prod_{k \in K} \Delta(C^k)$ is defined by (5.1) (when $\hat{C}^k = C^k$). By (5.1), for each $k \in K$, the mapping $(y^m)_{m \in M} \rightarrow x^k \times \prod_{k \in C^k} \left(\sum_{\xi^k \in C^k} y[\rho(\xi^k)] \right)$ is linear. Since ρ is injective and the payoffs in $G_0^I(y^I, x^K)$ depends affinely on each coordinate of $x^K = (x^k)_{k \in K}$, $G^I[y^I, y^M]$ depends affinely on each coordinate of $y^M = (y^m)_{m \in M}$. Note that $G\cdot$ then is induced by G in the same way $G_0\cdot$ is induced by G_0 ; by restricting the action profiles of the players in I , and setting $G[y](\cdot) = G^M(y, \cdot)$. This is shown in the following diagram:

Lemma 5.5. *Let $x \in \prod_{i \in I} \Delta(A^i)$, let $z \in \prod_{k \in K} \Delta(C^k)$ be an equilibrium of $G_0[x](\cdot)$, and let $v \in \prod_{m \in M} \Delta(D^m)$ be an equilibrium of $(\mathcal{S} \circ G_0[x])(\cdot) = G[x](\cdot)$ which generates z . Then (x, z) is an equilibrium of G_0 iff (x, v) is an equilibrium of G .*

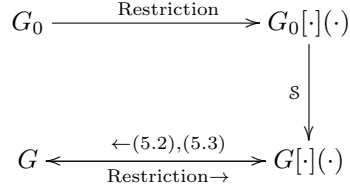


Figure 4: The Construction

Proof. It suffices to show that x is an equilibrium of $G_0^I(\cdot, z)$ iff and it is an equilibrium of $G^I(\cdot, v)$. This follows since (5.3) implies that $G_0^I(\cdot, z) = G^I(\cdot, v) \times \prod_{k \in C^k} (\sum_{\xi^k \in C^k} v[\rho(\xi^k)])$. \square

Now, we prove Theorem 5.5, let $\emptyset \neq X \subseteq \prod_{i \in I} \Delta(A^i)$ be compact and semi-algebraic. Let G_0 be a game on a set of binary players $I \cup K$ such that the projection of the Nash equilibria of G_0 to $\prod_i \Delta(A^i)$ is X . Such G_0 exists by Theorem 3.1. Apply the above construction, Theorem 5.6, and Lemma 5.5 to G_0 to obtain a game G on a set of players $I \cup M$, where $|M| = 3$, such that the projection of the Nash equilibria of G to $\prod_i \mathbb{R}^{A^i}$ is X .

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